HOSSAM GHANEM

(29) 8.8 *Improper Integrals (A)*

$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^{t} f(x)\,dx$	$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^{b} f(x)\,dx$
$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$	

If f continuous on $[\,a\,,b)$	$\int_a^{b} f(x)\,dx = \lim_{t\to b^-} \int_a^{t} f(x)\,dx$
If f continuous on $(\,a\,,b]$	$\int_a^{b} f(x)\,dx = \lim_{t\to a^+} \int_t^{b} f(x)\,dx$

The improper Integrals $\int_a^{b} f(x)\,dx$ Is convergent if $\int_a^{b} f(x)\,dx = L \in \mathcal{R}$	The improper Integrals $\int_a^{b} f(x)\,dx$ Is Divergent if $\int_a^{b} f(x)\,dx \to \begin{cases} \infty \\ -\infty \\ D.N.E \end{cases}$

Example 1*

52 July 24, 2010

(a) Use integration by parts to show that [2 marks]

$$\int \frac{(1-\ln x)}{x^2}\,dx = \frac{\ln x}{x} + C$$

(b) Determine the convergence or divergence of the improper integrals [$1\,\frac{1}{2}$ marks each]

(i) $\displaystyle\int_0^1 \frac{(1-\ln x)}{x^2}\,dx$ (ii) $\displaystyle\int_1^\infty \frac{(1-\ln x)}{x^2}\,dx$

Solution

$$I = \int \frac{(1-\ln x)}{x^2}\,dx$$

$u = 1 - \ln x$ $\qquad dv = \dfrac{1}{x^2}\,dx$

$du = -\dfrac{1}{x}\,dx$ $\qquad v = -\dfrac{1}{x}$

$$I = uv - \int v\,du$$

$$I = -\frac{1}{x}(1-\ln x) - \int \frac{1}{x^2}\,dx = -\frac{1}{x}(1-\ln x) + \frac{1}{x} + c = -\frac{1}{x} + \frac{\ln x}{x} + \frac{1}{x} + c = \frac{\ln x}{x} + c$$

$$I_1 = \int_0^1 \frac{(1-\ln x)}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{(1-\ln x)}{x^2}\,dx = \lim_{t\to 0^+} \left[\frac{\ln x}{x}\right]_t^1 = \lim_{t\to 0^+} 0 - \frac{\ln t}{t} = -\frac{-\infty}{0} = \infty$$

I_1 divergent

$$I_2 = \int_1^\infty \frac{(1-\ln x)}{x^2}\,dx = \lim_{t\to \infty} \int_1^t \frac{(1-\ln x)}{x^2}\,dx = \lim_{t\to \infty} \left[\frac{\ln x}{x}\right]_1^t = \lim_{t\to \infty} \frac{\ln t}{t} - 0 = \frac{\infty}{\infty}$$

$$\overset{LR}{=} \lim_{t\to\infty} \frac{\frac{1}{t}}{1} = 0$$

I_2 convergent

Example 2 *

40 August 7 , 2011

(4 pts) Determine whether the following improper integral is convergent or divergent , if convergent , find its value

$$\int_2^\infty \left(\frac{1}{\sqrt{x^2-1}} - \frac{1}{x-1}\right) dx$$

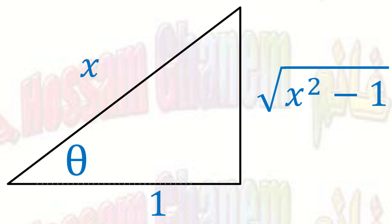
Solution

$$I_1 = \int \frac{1}{\sqrt{x^2-1}} dx$$

$x = \sec\theta \qquad dx = \sec\theta \tan\theta \ d\theta$

$\sec\theta = \frac{x}{1}$

$\theta = \sec^{-1} x$



$$I_1 = \int \frac{1}{\sqrt{\sec^2\theta - 1}} \sec\theta \tan\theta \ d\theta = \int \sec\theta \ d\theta = \ln|\sec\theta + \tan\theta| + c = \ln\left|x + \sqrt{x^2-1}\right| + c_1$$

$$I_2 = \int \frac{1}{x-1} dx = \ln|x-1| + c_2$$

$$I_3 = \int \left(\frac{1}{\sqrt{x^2-1}} - \frac{1}{x-1}\right) dx = \ln\left|x + \sqrt{x^2-1}\right| - \ln|x-1| + c_3 = \ln\left|\frac{x + \sqrt{x^2-1}}{x-1}\right| + c_3$$

$$I = \int_2^\infty \left(\frac{1}{\sqrt{x^2-1}} - \frac{1}{x-1}\right) dx = \lim_{t\to\infty} \int_2^t \left(\frac{1}{\sqrt{x^2-1}} - \frac{1}{x-1}\right) dx = \lim_{t\to\infty} \left[\ln\left|\frac{x + \sqrt{x^2-1}}{x-1}\right|\right]_2^t$$

$$= \lim_{t\to\infty} \left(\ln\left|\frac{t + \sqrt{t^2-1}}{t-1}\right| - \ln\left|\frac{2 + \sqrt{4-1}}{2-1}\right|\right) = 2 - \ln\left(2 + \sqrt{3}\right)$$

I convergent

Example 3 *

55 July 23 , 2011

Determine if the improper integral $\int_0^\infty \frac{dx}{e^{2x}+9}$ is convergent or divergent. Find its value if it is convergent. (4 pts.)

Solution

$$I_1 = \int \frac{1}{e^{2x}+9} dx = \int \frac{e^{-2x}}{1 + 9e^{-2x}} dx$$

Let $u = 1 + 9e^{-2x} \qquad du = -18e^{-2x} dx \qquad -\frac{1}{18} du = e^{-2x} dx$

$$I_1 = -\frac{1}{18} \int \frac{1}{u} du = -\frac{1}{18} \ln|u| + c = -\frac{1}{18} \ln|1 + 9e^{-2x}| + c$$

$$I = \int_0^\infty \frac{dx}{e^{2x}+9} = \lim_{t\to\infty} \int_0^t \frac{1}{e^{2x}+9} dx = \lim_{t\to\infty} \left[-\frac{1}{18} \ln|1 + 9e^{-2x}|\right]_0^t = -\frac{1}{18} \lim_{t\to\infty} (\ln|1 + 9e^{-2t}| - \ln 10)$$

$$= -\frac{1}{18} (\ln|1 + 0| - \ln 10) = \frac{1}{18} \ln 10$$

I convergent

Example 4

Determine if the improper integral $\displaystyle\int_{-\infty}^{0} \frac{e^x}{\sqrt{1+e^{-2x}}}\,dx$ is convergent or divergent. Find its value if it is convergent.

Solution

$$I_1 = \int \frac{e^x}{\sqrt{1+e^{-2x}}}\,dx = \int \frac{e^x \cdot e^x}{e^x\sqrt{1+e^{-2x}}}\,dx = \int \frac{e^{2x}}{\sqrt{e^{2x}+1}}\,dx$$

Let $u = e^{2x}+1 \qquad du = 2e^{2x}\,dx \qquad \frac{1}{2}du = e^{2x}\,dx$

$$I = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \sqrt{u} + c = \sqrt{e^{2x}+1} + c$$

$$I = \int_{-\infty}^{0} \frac{e^x}{\sqrt{1+e^{-2x}}}\,dx = \lim_{t\to-\infty}\int_{t}^{0} \frac{e^x}{\sqrt{1+e^{-2x}}}\,dx = \lim_{t\to-\infty}\left[\sqrt{e^{2x}+1}\right]_t^0 = \lim_{t\to-\infty}\left(\sqrt{2} - \sqrt{e^{2t}+1}\right)$$

$$= \sqrt{2} - \sqrt{0+1} = \sqrt{2} - 1$$

I convergent

Example 5

Determine if the improper integral $\displaystyle\int_{-\infty}^{1} \frac{1}{x^2+2x+5}\,dx$ is convergent or divergent. Find its value if it is convergent.

Solution

$$x^2+2x+5 = (x^2+2x+1)+5-1 = (x+1)^2+4$$

$$I_1 = \int \frac{1}{x^2+2x+5}\,dx = \int \frac{1}{(x+1)^2+4}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+1$$

$$I = \int_{-\infty}^{1} \frac{1}{x^2+2x+5}\,dx = \lim_{t\to-\infty}\int_{t}^{1} \frac{1}{x^2+2x+5}\,dx = \lim_{t\to-\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)\right]_t^1 = \frac{1}{2}\lim_{t\to-\infty}\left[\frac{\pi}{4} - \tan^{-1}\left(\frac{t+1}{2}\right)\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - \frac{-\pi}{2}\right) = \frac{1}{2}\cdot\frac{3\pi}{4} = \frac{3\pi}{8}$$

I convergent

Example 6

Determine if the improper integral $\displaystyle\int_{1}^{4} \frac{1}{\sqrt{4x-x^2}}\,dx$ is convergent or divergent. Find its value if it is convergent.

Solution

$$4x - x^2 = 4 - (x^2-4x+4) = 4-(x-2)^2$$

$$I_1 = \int \frac{1}{\sqrt{4x-x^2}}\,dx = \int \frac{1}{\sqrt{4-(x-2)^2}}\,dx = \sin^{-1}\left(\frac{x-2}{2}\right)+c$$

$$I_1 = \int_{1}^{4} \frac{1}{\sqrt{4x-x^2}}\,dx = \lim_{t\to4^-}\int_{1}^{t} \frac{1}{\sqrt{4x-x^2}}\,dx = \lim_{t\to4^-}\left[\sin^{-1}\left(\frac{x-2}{2}\right)\right]_1^t = \lim_{t\to4^-}\left[\sin^{-1}\left(\frac{t-2}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right]$$

$$= \sin^{-1}1 + \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}+\frac{\pi}{6} = \frac{3\pi+\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

I convergent

Example 7 Determine if the improper integral $\int_0^1 \frac{1}{\sqrt[3]{1-x}}dx$ is convergent or divergent. Find its value if it is convergent.

Solution

$$I_1 = \int \frac{1}{\sqrt[3]{1-x}}dx = \int (1-x)^{-\frac{1}{3}}dx = \frac{-3}{2}(1-x)^{\frac{2}{3}}$$

$$I_1 = \int_0^1 \frac{1}{\sqrt[3]{1-x}}dx = \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt[3]{1-x}}dx = \lim_{t\to 1^-}\left[\frac{-3}{2}(1-x)^{\frac{2}{3}}\right]_0^t = -\frac{3}{2}\lim_{t\to 1^-}\left[(1-t)^{\frac{2}{3}}-1\right] = -\frac{3}{2}(0-1) = \frac{3}{2}$$

I convergent

Homework

Determine whether the following improper integrals convergent or divergent , and find its value if it convergent

1	$\int_0^{\frac{\pi}{2}} \frac{1}{1-\cos x}\,dx$	9	$\int_1^{\sqrt{2}} \operatorname{csch}(\ln x)\,dx$	17	$\int_0^{\pi} \frac{1}{1+\cos x+\sin x}\,dx$
2	$\int_0^{\infty} \frac{x}{x^4+1}\,dx$	10	$\int_0^{1} \ln(x+x^2)\,dx$	18	$\int_1^{10} \frac{x}{\sqrt[3]{x-2}}\,dx$
3	$\int_1^{\infty} \frac{x^3}{1+x^8}\,dx$	11	$\int_0^{\infty} \frac{x}{e^x}\,dx$	19	$\int_0^{1} \frac{1}{\sqrt[3]{1-x}}\,dx$
4	$\int_0^{\infty} \frac{x^2}{1+x^6}\,dx$	12	$\int_0^{\infty} xe^{-x}\,dx$	20	$\int_0^{2} \frac{dx}{\sqrt{-x^2+4x-3}}$
5	$\int_1^{\infty} \frac{dx}{x\sqrt{6x-1}}$	13	$\int_{-\infty}^{1} \frac{1}{x^2+2x+5}\,dx$	21	$\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx$
6	$\int_3^{\infty} \frac{1}{\sqrt{x}\,(9+x)}\,dx$	14	$\int_2^{\infty} \frac{dx}{x\sqrt{x^2+4}}$	22	$\int_0^{\infty} \frac{1}{1+e^{-x}}\,dx$
7	$\int_0^{1} \frac{1}{x^{\frac{3}{2}}+x^{\frac{1}{2}}}\,dx$	15	$\int_1^{\infty} \frac{dx}{x\sqrt{x^2+4}}$	23	$\int_{\ln 3}^{\infty} \frac{e^x}{e^{2x}-3e^x+2}\,dx$
8	$\int_0^{1} \frac{e^{(\sqrt[3]{x})}}{\sqrt[3]{x^2}}\,dx$	16	$\int_0^{\infty} \frac{1}{x(1+\ln x)^2}\,dx$	24	$\int_1^{\infty} \frac{\ln(1+x^2)}{x^2}\,dx$

Homework

Determine whether the following improper integrals convergent or divergent , and find its value if it convergent

25	$\int_{-1}^{1} \frac{1+x}{1+\sqrt[3]{x}}\,dx$	31	$\int_{1}^{3} \frac{1}{\sqrt{3x-x^2}}\,dx$	37	$\int_{0}^{2} (\ln x)^2\,dx$
26	$\int_{0}^{1} \frac{1}{x^{\frac{3}{2}}+x^{\frac{1}{2}}}\,dx$	32	$\int_{0}^{\infty} e^{-\sqrt{x}}\,dx$	38	$\int_{0}^{1} x\ln x\,dx$
27	$\int_{0}^{1} \frac{1}{x+\sqrt{x}}\,dx$	33	$\int_{0}^{\infty} (5-x)e^{-2x}\,dx$	39	$\int_{0}^{1} \ln(x^x)\,dx$
28	$\int_{2}^{\infty} \frac{dx}{x\sqrt{x^2+4}}$	34	$\int_{-\infty}^{0} \frac{e^x}{\sqrt{1+e^{-2x}}}\,dx$	40	$\int_{1}^{\infty} \frac{\ln x}{x^2}\,dx$
29	$\int_{0}^{2} \frac{dx}{\sqrt{2x-x^2}}$	35	$\int_{0}^{\infty} \frac{e^{-x}}{e^x+1}\,dx$	41	$\int_{0}^{1} \frac{\ln x}{\sqrt{x}}\,dx$
30	$\int_{-3}^{2} \frac{dx}{\sqrt{3-2x-x^2}}$	33	$\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx$	42	$\int_{0}^{1} \frac{1}{x(\ln^2 x+4)}\,dx$